On the determinant formulas by Borodin, Okounkov, Baik, Deift, and Rains

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We give alternative proofs to (block case versions of) some formulas for Toeplitz and Fredholm determinants established recently by the authors of the title. Our proof of the Borodin-Okounkov formula is very short and direct. The proof of the Baik-Deift-Rains formulas is based on standard manipulations with Wiener-Hopf factorizations.

1. The formulas

Let $\mathbf{T}$ be the complex unit circle and let $L^\infty := L^\infty_{N\times N}$ stand for the algebra of all $N \times N$ matrix functions with entries in $L^\infty(\mathbf{T})$. Given $a \in L^\infty$, we denote by $\{a_k\}_{k\in\mathbf{Z}}$ the sequence of the Fourier coefficients,

$$a_k = \frac{1}{2\pi}\int_0^{2\pi} a(e^{i\theta})e^{-ik\theta}d\theta = \frac{1}{2\pi i}\int_{\mathbf{T}} a(z)z^{-k}\frac{dz}{z}.$$

The matrix function a generates several structured (block) matrices:

$$\begin{aligned}
T(a) &= (a_{j-k})_{j,k=0}^{\infty} \quad \text{(infinite block Toeplitz)},\\
T_n(a) &= (a_{j-k})_{j,k=0}^{n-1} \quad \text{(finite block Toeplitz)},\\
H(a) &= (a_{j+k+1})_{j,k=0}^{\infty} \quad \text{(block Hankel)},\\
H(\tilde{a}) &= (a_{-j-k-1})_{j,k=0}^{\infty} \quad \text{(block Hankel)},\\
L(a) &= (a_{j-k})_{j,k=-\infty}^{\infty} \quad \text{(block Laurent)},\\
L(\tilde{a}) &= (a_{k-j})_{j,k=-\infty}^{\infty} \quad \text{(block Laurent)}.
\end{aligned}$$

The matrices $T(a), H(a), H(\tilde{a})$ induce bounded operators on $\ell^2(\mathbf{Z}_+, \mathbf{C}^N)$, and the matrices $L(a), L(\tilde{a})$ define bounded operators on $\ell^2(\mathbf{Z}, \mathbf{C}^N)$.

Let $\|\cdot\|$ be any matrix norm on $\mathbf{C}^{N\times N}$. We need the following classes of matrix functions:

$$W = \{a \in L^\infty : \sum_{n\in\mathbf{Z}} \|a_n\| < \infty\} \quad \text{(Wiener algebra)},$$

$$K_1^1 = \{a \in L^\infty : \sum_{n\in\mathbf{Z}} (|n|+1)\|a_n\| < \infty\} \quad \text{(weighted Wiener algebra)},$$

$$K_2^{1/2} = \{a \in L^\infty : \sum_{n\in\mathbf{Z}} (|n|+1)\|a_n\|^2 < \infty\} \quad \text{(Krein algebra)},$$

$$H_\pm^\infty = \{a \in L^\infty : a_{\mp n} = 0 \ \text{ for } \ n > 0\} \quad \text{(Hardy space)}.$$

Clearly, $K_1^1 \subset K_2^{1/2}$. Given a subset E of L^∞, we say that a matrix function $a \in L^\infty$ has a right (resp. left) canonical Wiener-Hopf factorization in E and write $a \in \Phi_r(E)$ (resp. $a \in \Phi_l(E)$) if a can be represented in the form $a = u_- u_+$ (resp. $a = v_+ v_-$) with

$$u_-, v_-, u_-^{-1}, v_-^{-1} \in E \cap H_-^\infty, \quad u_+, v_+, u_+^{-1}, v_+^{-1} \in E \cap H_+^\infty.$$

It is well known (see, e.g., [5], [7]) that if $a \in \Phi_r(L^\infty)$ then $T(a)$ is invertible and $T^{-1}(a) = T(u_+^{-1})T(u_-^{-1})$ and that for $a \in K_1^1$ (resp. $a \in W \cap K_2^{1/2}$) we have

$$a \in \Phi_r(K_1^1) \quad \text{(resp. } a \in \Phi_r(K_2^{1/2})) \quad \Longleftrightarrow \quad T(a) \ \text{ is invertible.}$$

If $a \in K_1^1$ then $H(a)$ and $H(\tilde{a})$ are trace class operators, and if $a \in K_2^{1/2}$, then $H(a)$ and $H(\tilde{a})$ are Hilbert-Schmidt.

We define the projections P, Q, Q_n $(n \in \mathbf{Z})$ on the space $\ell^2(\mathbf{Z}, \mathbf{C}^N)$ by

$$(Px)_k = \begin{cases} x_k & \text{for} \quad k \geq 0, \\ 0 & \text{for} \quad k < 0, \end{cases} \qquad (Qx)_k = \begin{cases} 0 & \text{for} \quad k \geq 0, \\ x_k & \text{for} \quad k < 0, \end{cases}$$

$$(Q_n x)_k = \begin{cases} x_k & \text{for} \quad k \geq n, \\ 0 & \text{for} \quad k < n. \end{cases}$$

For $n \geq 1$, we let P_n denote the projection on $\ell^2(\mathbf{Z}_+, \mathbf{C}^N)$ given by

$$(P_n x)_k = \begin{cases} x_k & \text{for} \quad 0 \leq k \leq n-1, \\ 0 & \text{for} \quad k \geq n. \end{cases}$$

If $n \geq 0$, we can also think of Q_n as an operator on $\ell^2(\mathbf{Z}_+, \mathbf{C}^N)$. Note that the notation used here differs from the one of [1], but that our notation is standard in the Toeplitz business.

On defining the flip operator J on $\ell^2(\mathbf{Z}, \mathbf{C}^N)$ by $(Jx)_k = x_{-k-1}$, we can write

$$T(a) = PL(a)P|\mathrm{Im}\, P, \quad H(a) = PL(a)QJ|\mathrm{Im}\, P, \quad H(\tilde{a}) = JQL(a)P|\mathrm{Im}\, P \tag{1}$$

Moreover, we may identify the operator $L(a)$ on $\ell^2(\mathbf{Z}, \mathbf{C}^N)$ with the operator of multiplication by a on $L^2(\mathbf{T}, \mathbf{C}^N)$. Since P, Q, J are also naturally defined on the space $L^2(\mathbf{T}, \mathbf{C}^N)$, formulas (1) enable us to interpret Toeplitz and Hankel operators as operators on the Hardy space $H^2(\mathbf{T}, \mathbf{C}^N)$.

For $a \in \Phi_l(L^\infty)$, the geometric mean $G(a)$ is defined by $G(a) = (\det v_+)_0 (\det v_-)_0$, where $(\cdot)_k$ stands for the kth Fourier coefficient. Thus, with an appropriately chosen logarithm,

$$G(a) = \exp(\log \det a)_0.$$

Let now $a \in \Phi_r(K_2^{1/2}) \cap \Phi_l(K_2^{1/2})$ and let $a = u_- u_+$ and $a = v_+ v_-$ be canonical Wiener-Hopf factorizations. Put $b = v_- u_+^{-1}$ and $c = u_-^{-1} v_+$. Obviously, $bc = I$. Using (1) it is easily seen that

$$T(b)T(c) + H(b)H(\tilde{c}) = I. \tag{2}$$

Since Hankel operators generated by matrix functions in $K_2^{1/2}$ are Hilbert-Schmidt, the operator $H(b)H(\tilde{c})$ is in the trace class. From (2) we infer that $I - H(b)H(\tilde{c})$ is invertible. We put

$$E(a) = 1/\det(I - H(b)H(\tilde{c})).$$

One can show (again see [5], [7]) that $E(a) = \det T(a)T(a^{-1})$ and that in the scalar case ($N = 1$) we also have

$$E(a) = \exp \sum_{k=1}^{\infty} k (\log a)_k (\log a)_{-k}.$$

Theorem 1.1 (Borodin-Okounkov à la Widom). *If $a \in \Phi_r(K_2^{1/2}) \cap \Phi_l(K_2^{1/2})$ then*

$$\det T_n(a) = G(a)^n E(a) \det(I - Q_n H(b) H(\tilde{c}) Q_n) \tag{3}$$

for all $n \geq 1$.

In the scalar case, this beautiful theorem was established by Borodin and Okounkov in [3]. It answered a question raised by Its and Deift. The proof of [3] is rather complicated. Three simpler proofs were subsequently found by Basor and Widom [2] (who also extended the theorem to the block case) and by the author [4]. We here give still another proof, which is very short and direct.

Now suppose that $a \in \Phi_r(K_1^1) \cap \Phi_l(K_1^1)$ ($\subset \Phi_r(K_2^{1/2}) \cap \Phi_l(K_2^{1/2})$). Define b and c as above. We have

$$P - L(c)Q_n L(b) = (PL(c) - L(c)Q_n)L(b) = (PL(c)Q - QL(c)P + L(c)(P - Q_n))L(b)$$

and since $PL(c)Q$ and $QL(c)P$ are trace class operators (notice that $b, c \in K_1^1$) and the operator $P - Q_n$ has finite rank, we see that $P - L(c)Q_n L(b)$ is trace class.

Theorem 1.2 (Baik-Deift-Rains). *If $a \in \Phi_r(K_1^1) \cap \Phi_l(K_1^1)$ then*

$$\det T_n(a) = G(a)^n E(a) 2^{-nN} \det(I + P - L(c)Q_n L(b)) \tag{4}$$

for all $n \geq 1$.

Clearly, to prove Theorem 1.2 it suffices to prove Theorem 1.1 and to verify that

$$\det(I + P - L(c)Q_n L(b)) = 2^{nN} \det(I - Q_n H(b) H(\tilde{c}) Q_n) \tag{5}$$

for all $n \geq 1$. By virtue of (1),

$$\det(I - Q_n H(b)H(\tilde{c})Q_n) = \det(I - Q_n L(b)QL(c)Q_n)$$

for all $n \geq 1$. The right-hand side of the last equality makes sense for all $n \in \mathbf{Z}$. In fact, we have the following generalization of (5).

Theorem 1.3 (Baik-Deift-Rains). *If $a \in \Phi_r(K_1^1) \cap \Phi_l(K_1^1)$ then for all $n \in \mathbf{Z}$,*

$$\begin{aligned} &\det(I + sP - sL(a)Q_nL(a^{-1})) \\ &\quad = (1+s)^{nN} \det(I - s^2 Q_n L(a^{-1})QL(a)Q_n) \quad (s \neq -1) \qquad (6) \\ &\quad = (1-s)^{-nN} \det(I - s^2 (I-Q_n)L(a^{-1})PL(a)(I-Q_n)) \quad (s \neq 1) \qquad (7) \end{aligned}$$

Theorems 1.2 and 1.3 are in [1]. The proof given there is as follows: the formulas are easily seen if some operator that is no trace class operator were a trace class operator and to save that insight the authors employ an approximation argument. We here present a proof that is a little more direct and uses Wiener-Hopf factorization.

Theorem 1.1 is proved in Section 2, the proofs of Theorems 1.2 and 1.3 are given in Section 3. In Section 4 we relax the hypothesis of Theorem 1.3 to the requirement that a be in K_1^1 and that $\det a$ have no zeros on the unit circle, and in Section 5 we prove a "multi-interval" version of Theorem 1.3.

2. Proof of the Borodin-Okounkov formula

If K is an arbitrary trace class operator on $\ell^2(\mathbf{Z}_+, \mathbf{C}^N)$ and $I - K$ is invertible, then

$$\det P_n(I-K)^{-1}P_n = \frac{\det(I - Q_nKQ_n)}{\det(I-K)}. \qquad (8)$$

With K replaced by P_mKP_m, this is Jacobi's theorem on the principle $n \times n$ minor of the inverse of a (finite) matrix. In the general case the identity follows from the fact that P_mKP_m converges to K in the trace norm as $m \to \infty$. For $K = H(b)H(\tilde{c})$ we obtain from (2) that

$$\begin{aligned} P_n(I-K)^{-1}P_n &= P_nT^{-1}(c)T^{-1}(b)P_n \\ &= P_nT(v_+^{-1})T(u_-)T(u_+)T(v_-^{-1})P_n = T_n(v_+^{-1})T_n(a)T_n(v_-^{-1}), \end{aligned}$$

and since $\det T_n(v_+^{-1})T_n(a)T_n(v_-^{-1}) = G(a)^{-n} \det T_n(a)$, we get (3) from (8). ■

3. Proof of the Baik-Deift-Rains formulas

In what follows we abbreviate $L(a)$ to a. Equivalently, we may regard all operators on L^2 instead of ℓ^2 and may therefore think of a as multiplication by a. Notice that if $a \in K_1^1$ is invertible in L^∞, then a^{-1} also belongs to K_1^1.

Lemma 3.1. *If a and a^{-1} are in K_1^1 then*

$$P - aQ_na^{-1}, \quad Q_na^{-1}QaQ_n, \quad (I-Q_n)a^{-1}Pa(I-Q_n)$$

are trace class operators for all $n \in \mathbf{Z}$.

Proof. We have

$$\begin{aligned}
&P - aQ_na^{-1} = (Pa - aQ_n)a^{-1} = (PaQ - QaP + a(P - Q_n))a^{-1},\\
&Q_na^{-1}QaQ_n = -Q_na^{-1}QaP + Q_na^{-1}Qa(P - Q_n),\\
&(I - Q_n)a^{-1}Pa(I - Q_n) = (I - Q_n)a^{-1}PaQ + (I - Q_n)a^{-1}Pa(P - Q_n),
\end{aligned}$$

and since PaQ and QaP are trace class and $P - Q_n$ has finite rank, we arrive at the assertion. ■

We put

$$f_n(s) = \det(I + sP - saQ_na^{-1}).$$

Proposition 3.2. *If $a \in \Phi_r(K_1^1)$ and $n \geq 0$, then*

$$f_n(s) = (1 + s)^{nN} \det(I - s^2Q_na^{-1}QaQ_n). \tag{9}$$

Proof. Let $a = u_-u_+$ be a right canonical Wiener-Hopf factorization in K_1^1. Then

$$f_n(s) = \det(I + sP - su_-u_+Q_nu_+^{-1}u_-^{-1}) = \det(I + su_-^{-1}Pu_- - su_+Q_nu_+^{-1}),$$

and since $u_-^{-1}Pu_- = P + Qu_-^{-1}Pu_-P$ and $u_+Q_n = Pu_+Q_n$, we get

$$f_n(s) = \det(I + sQu_-^{-1}Pu_-P + sP - sPu_+Q_nu_+^{-1}).$$

The operator $I + sQu_-^{-1}Pu_-P$ has the inverse $I - sQu_-^{-1}Pu_-P$ and its determinant is 1. Hence,

$$\begin{aligned}
f_n(s) &= \det(I + (sP - sPu_+Q_nu_+^{-1})(I - sQu_-^{-1}Pu_-P))\\
&= \det(I + sP - sPu_+Q_nu_+^{-1} + s^2Pu_+Q_nu_+^{-1}Qu_-^{-1}Pu_-P).
\end{aligned}$$

Because $\det(I + PA) = \det(I + PAP)$ and

$$Pu_-^{\pm 1} = Pu_-^{\pm 1}P, \quad u_+^{\pm 1}P = Pu_+^{\pm 1}P, \quad u_-^{\pm 1}Q = Qu_-^{\pm 1}Q, \quad Qu_+^{\pm 1} = Qu_+^{\pm 1}Q,$$

it follows that

$$\begin{aligned}
f_n(s) &= \det(I + sP - sPu_+Q_nu_+^{-1}P + s^2Pu_+Q_nu_+^{-1}Qu_-^{-1}Pu_-P)\\
&= \det(I + sP - sPu_+Q_nu_+^{-1}P - s^2Pu_+Q_nu_+^{-1}Qu_-^{-1}Qu_-P)\\
&= \det(I + sP - sQ_n - s^2Pu_+^{-1}Pu_+Q_nu_+^{-1}Qu_-^{-1}Qu_-Pu_+P)\\
&= \det(I + sP - sQ_n - s^2Q_nu_+^{-1}Qu_-^{-1}Qu_-u_+P)\\
&= \det(I + sP - sQ_n - s^2Q_nu_+^{-1}u_-^{-1}Qu_-u_+P)\\
&= \det(I + sP - sQ_n - s^2Q_na^{-1}QaP)\\
&= \det(I + sP - sQ_n)\det(I - s^2Q_na^{-1}QaP)\\
&= (1 + s)^{nN}\det(I - s^2Q_na^{-1}QaQ_n) \quad ■
\end{aligned}$$

At this point we have proved formula (6) for $n \geq 0$ and thus formula (5) and Theorem 1.2. We are left with switching from (6) to (7) and passing to negative n's.

Proposition 3.3. *If $a \in \Phi_l(K_1^1)$ and $n \geq 0$, then*

$$f_{-n}(s) = (1-s)^{nN} \det(I - s^2(I - Q_{-n})a^{-1}Pa(I - Q_{-n})). \tag{10}$$

Proof. We repeat the argument of the preceding proof, but now we work with the left canonical Wiener-Hopf factorization $a = v_+v_-$. We have

$$\begin{aligned}
f_{-n}(s) &= \det(I + sP - saQ_{-n}a^{-1}) \\
&= \det(I - sQ + sa(I - Q_{-n})a^{-1}) \\
&= \det(I - sQ + sv_+v_-(I - Q_{-n})v_-^{-1}v_+^{-1}) \\
&= \det(I - sv_+^{-1}Qv_+ + sv_-(I - Q_{-n})v_-^{-1}) \\
&= \det(I - sPv_+^{-1}Qv_+Q - sQ + sv_-(I - Q_{-n})v_-^{-1}) \\
&= \det(I + (-sQ + sQv_-(I - Q_{-n})v_-^{-1})(I + sPv_+^{-1}Qv_+Q)) \\
&= \det(I - sQ + sQv_-(I - Q_{-n})v_-^{-1}Q + s^2Qv_-(I - Q_{-n})v_-^{-1}Pv_+^{-1}Pv_+Q) \\
&= \det(I - sQ + s(I - Q_{-n}) - s^2Qv_-^{-1}Qv_-(I - Q_{-n})v_-^{-1}Pv_+^{-1}Pv_+Qv_-Q) \\
&= \det(I - sQ + s(I - Q_{-n}) - s^2(I - Q_{-n})v_-^{-1}v_+^{-1}Pv_+v_-Q) \\
&= \det(I - sQ + s(I - Q_{-n}) - s^2(I - Q_{-n})a^{-1}PaQ) \\
&= \det(I - sQ + s(I - Q_{-n}))\det(I - s^2(I - Q_{-n})a^{-1}PaQ) \\
&= (1-s)^{nN}\det(I - s^2(I - Q_{-n})a^{-1}Pa(I - Q_{-n})). \quad \blacksquare
\end{aligned}$$

Lemma 3.4. *If a and a^{-1} are in K_1^1 and $n \in \mathbf{Z}$, then*

$$f_n(-s)f_n(s) = \det(I - s^2(I - Q_n)a^{-1}Pa(I - Q_n))\det(I - s^2Q_na^{-1}QaQ_n). \tag{11}$$

Proof. We have

$$\begin{aligned}
&(I - sP + saQ_na^{-1})(I + sP - saQ_na^{-1}) \\
&\quad = I - s^2P + s^2PaQ_na^{-1}P - s^2QaQ_na^{-1}Q \\
&\quad = I - s^2Pa(I - Q_n)a^{-1}P - s^2QaQ_na^{-1}Q.
\end{aligned}$$

Taking determinants we obtain that

$$\begin{aligned}
f_n(-s)f_n(s) &= \det(I - s^2Pa(I - Q_n)a^{-1}P)\det(I - s^2QaQ_na^{-1}Q) \\
&= \det(I - s^2(I - Q_n)a^{-1}Pa(I - Q_n))\det(I - s^2Q_na^{-1}QaQ_n). \quad \blacksquare
\end{aligned}$$

Proposition 3.5. *If $a \in \Phi_l(K_1^1)$ and $n \geq 0$, then*

$$(1+s)^{nN}f_{-n}(s) = \det(I - s^2Q_{-n}a^{-1}QaQ_{-n}), \tag{12}$$

and if $a \in \Phi_r(K_1^1)$ and $n \geq 0$, then

$$f_n(s) = (1-s)^{nN}\det(I - s^2(I - Q_n)a^{-1}Pa(I - Q_n)). \tag{13}$$

Proof. Proposition 3.3 and Lemma 3.4 give

$$\begin{aligned}
&f_{-n}(s)(1+s)^{nN}\det(I - s^2(I - Q_{-n})a^{-1}Pa(I - Q_{-n})) = f_{-n}(s)f_{-n}(-s) \\
&\quad = \det(I - s^2(I - Q_{-n})a^{-1}Pa(I - Q_{-n}))\det(I - s^2Q_{-n}a^{-1}QaQ_{-n}).
\end{aligned}$$

Since $\det(I - s^2(I - Q_{-n})a^{-1}Pa(I - Q_{-n})) \neq 0$ for sufficiently small s, we get (12) for these s and then by analytic continuation for all s. Analogously, using Proposition 3.2 and Lemma 3.3 we get

$$
\begin{aligned}
&f_n(s)(1-s)^{nN} \det(I - s^2 Q_n a^{-1} Q a Q_n) = f_n(s) f_n(-s) \\
&\quad = \det(I - s^2(I - Q_n)a^{-1}Pa(I - Q_n)) \det(I - s^2 Q_n a^{-1} Q a Q_n),
\end{aligned}
$$

which implies (13). ■

Theorem 1.3 is the union of Propositions 3.2, 3.4, and 3.5.

4. Non-invertible operators

The hypothesis of Theorem 1.3 is that a be in $\Phi_r(K_1^1) \cap \Phi_l(K_1^1)$, which is equivalent to the invertibility of both $T(a)$ and $T(a^{-1})$. The theorem of this section, which is also from [1], relaxes this hypothesis essentially: we only require that $T(a)$ be Fredholm (which automatically implies that $T(a^{-1})$ is also Fredholm). Notice that if a is continuous (and matrix functions in K_1^1 are continuous) then $T(a)$ is a Fredholm operator if and only if $\det a$ has no zeros on $\mathbf{T}$. In that case the index of $T(a)$ is minus the winding number of $\det a$, $\operatorname{Ind} T(a) = -\operatorname{wind} \det a$.

Lemma 4.1. *If $a \in K_1^1$ and $T(a)$ is Fredholm of index zero, then* (6) *and* (7) *are valid.*

Proof. A theorem by Widom [6] tells us that there exist a trigonometric polynomial φ and a number $\varrho > 0$ such that $T(a + \varepsilon\varphi)$ is invertible for all complex numbers ε satisfying $0 < |\varepsilon| < \varrho$. Since $T(a + \varepsilon\varphi)$ is invertible, we conclude that $a + \varepsilon\varphi \in \Phi_r(K_1^1)$. Thus, (9) and (13) are true with a replaced by $a + \varepsilon\varphi$. From the proof of Lemma 3.1 we see that

$$
\begin{aligned}
&L(a + \varepsilon\varphi) Q_n L((a + \varepsilon\varphi)^{-1}) \to L(a) Q_n L(a^{-1}), \\
&Q_n L((a + \varepsilon\varphi)^{-1}) Q L(a + \varepsilon\varphi) Q_n \to Q_n L(a^{-1}) Q L(a) Q_n
\end{aligned}
$$

in the trace norm as $\varepsilon \to 0$. This gives (9) and (13). The proof of formulas (10) and (12) is analogous. ■

Lemma 4.2. *If the scalar-valued function $a \in K_1^1$ has no zeros on the unit circle and winding number w about the origin, then for all $n \in \mathbf{Z}$,*

$$
\begin{aligned}
&\det(I + sP - sL(a)Q_n L(a^{-1})) \\
&\quad = (1+s)^{n+w} \det(I - s^2 Q_n L(a^{-1}) Q L(a) Q_n) \quad (s \neq -1), \qquad (14) \\
&\det(I + sP - sL(a)Q_n L(a^{-1})) \\
&\quad = (1-s)^{-n-w} \det(I - s^2 (I - Q_n) L(a^{-1}) P L(a) (I - Q_n)) \quad (s \neq 1). \qquad (15)
\end{aligned}
$$

Proof. Recall that χ_w is defined by $\chi_w(t) = t^w$. We can write $a = \chi_w b$ with $\operatorname{wind} b = 0$. The key observation is that $\chi_w Q_n \chi_{-w} = Q_{n+w}$. Consequently,

$$
\begin{aligned}
&\det(I + sP - saQ_n a^{-1}) \\
&\quad = \det(I + sP - sb\chi_w Q_n \chi_{-w} b^{-1}) \\
&\quad = \det(I + sP - sbQ_{n+w} b^{-1})
\end{aligned}
$$

$$
\begin{aligned}
&= (1+s)^{n+w} \det(I - s^2 Q_{n+w} b^{-1} Q b Q_{n+w}) \quad \text{(by Theorem 1.3)} \\
&= (1+s)^{n+w} \det(I - s^2 \chi_w Q_n \chi_{-w} b^{-1} Q b \chi_w Q_n \chi_{-w}) \\
&= (1+s)^{n+w} \det(I - s^2 Q_n a^{-1} Q a Q_n),
\end{aligned}
$$

which is (14). Analogously one can derive (15) from Theorem 1.3. ■

Theorem 4.3 (Baik-Deift-Rains). *Let a be an $N\times N$ matrix function in K_1^1 and suppose $\det a$ has no zeros on $\mathbf{T}$. Put $w = \operatorname{wind} \det a$. Then for all $n \in \mathbf{Z}$,*

$$
\begin{aligned}
&\det(I + sP - sL(a)Q_n L(a^{-1})) \\
&\quad = (1+s)^{nN+w} \det(I - s^2 Q_n L(a^{-1}) Q L(a) Q_n) \quad (s \neq -1), \qquad (16) \\
&\det(I + sP - sL(a)Q_n L(a^{-1})) \\
&\quad = (1-s)^{-nN-w} \det(I - s^2 (I-Q_n) L(a^{-1}) P L(a) (I-Q_n)) \quad (s \neq 1). \qquad (17)
\end{aligned}
$$

Proof (*after Percy Deift*). We extend a to an $(N+1)\times(N+1)$ matrix function c by adding the $N+1, N+1$ entry χ_{-w}:

$$
c = \begin{pmatrix} a & 0 \\ 0 & \chi_{-w} \end{pmatrix}.
$$

Since $T(c)$ is Fredholm of index zero, we deduce from Lemma 4.1 that

$$
\det(I + sP - scQ_n c^{-1}) = (1+s)^{n(N+1)} \det(I - s^2 Q_n c^{-1} Q c Q_n). \qquad (18)
$$

Obviously,

$$
\begin{aligned}
\det(I + sP - scQ_n c^{-1}) &= \det(I + sP - saQ_n a^{-1}) \det(I + sP - s\chi_{-w} Q_n \chi_w), \qquad (19) \\
\det(I - s^2 Q_n c^{-1} Q c Q_n) &= \det(I - s^2 Q_n a^{-1} Q a Q_n) \det(I - s^2 Q_n \chi_w Q \chi_{-w} Q_n). \qquad (20)
\end{aligned}
$$

Lemma 4.2 implies that

$$
\det(I + sP - s\chi_{-w} Q_n \chi_w) = (1+s)^{n-w} \det(I - s^2 Q_n \chi_w Q \chi_{-w} Q_n) \qquad (21)
$$

(which, by the way, can also be verified straightforwardly in the particular case at hand). Combining (18), (19), (20), (21) we arrive at (16). The proof of (17) is analogous. ■

5. The multi-interval case

The purpose of this section is to show that the argument employed in Section 3 also works in the so-called multi-interval case. The following theorem is again from [1].

Theorem 5 (Baik-Deift-Rains). *Let $0 = n_0 \le n_1 \le \ldots \le n_k \le n_{k+1} = \infty$ be integers and let $s_1, \ldots, s_k$ be complex numbers such that $s_k - s_j \neq -1$ for all j. Put $s_0 = 0$. If $a \in \Phi_r(K_1^1)$ then*

$$
\begin{aligned}
&\det\left(I + \sum_{j=1}^{k} (s_j - s_{j-1})(P - L(a)Q_{n_j} L(a^{-1}))\right) \\
&= \left(\prod_{j=0}^{k-1} (1+s_k-s_j)^{n_{j+1}-n_j}\right) \det\left(I - \left(\sum_{j=1}^{k} \frac{s_k s_j}{1+s_k-s_j} P_{[n_j,n_{j+1})}\right) L(a^{-1}) Q L(a)\right), \qquad (22)
\end{aligned}
$$

where $P_{[n_j,n_{j+1})} = Q_{n_j} - Q_{n_{j+1}}$ is the projection onto the coordinates l with $n_j \le l < n_{j+1}$.

Proof. Proceeding exactly as in the proof of Proposition 3.2 we get

$$\begin{aligned}
&\det\left(I+\sum_{j=1}^{k}(s_j-s_{j-1})(P-aQ_{n_j}a^{-1})\right)\\
&\quad=\det\left(I+\sum_{j=1}^{k}\Big((s_j-s_{j-1})u_-^{-1}Pu_- -(s_j-s_{j-1})u_+Q_{n_j}u_+^{-1})\Big)\right)\\
&\quad=\det\left(I+\sum_{j=1}^{k}\Big((s_j-s_{j-1})Qu_-^{-1}Pu_-P+(s_j-s_{j-1})P-(s_j-s_{j-1})Pu_+Q_{n_j}u_+^{-1})\Big)\right)\\
&\quad=\det\left(I+\left(\sum_{j=1}^{k}(s_j-s_{j-1})P-\sum_{j=1}^{k}(s_j-s_{j-1})Pu_+Q_{n_j}u_+^{-1}\right)\right.\\
&\qquad\qquad\left.\times\left(I-\sum_{l=1}^{k}(s_l-s_{l-1})Qu_-^{-1}Pu_-P\right)\right)\\
&\quad=\det\left(I+\sum_{j=1}^{k}(s_j-s_{j-1})P-\sum_{j=1}^{k}(s_j-s_{j-1})Pu_+Q_{n_j}u_+^{-1}\right.\\
&\qquad\qquad\left.+\sum_{j,l}(s_j-s_{j-1})(s_l-s_{l-1})Pu_+Q_{n_j}u_+^{-1}Qu_-^{-1}Pu_-P\right)\\
&\quad=\det\left(I+\sum_{j=1}^{k}(s_j-s_{j-1})P-\sum_{j=1}^{k}(s_j-s_{j-1})Pu_+Q_{n_j}u_+^{-1}P\right.\\
&\qquad\qquad\left.-\sum_{j,l}(s_j-s_{j-1})(s_l-s_{l-1})Pu_+Q_{n_j}u_+^{-1}Qu_-^{-1}Qu_-P\right)\\
&\quad=\det\left(I+\sum_{j=1}^{k}(s_j-s_{j-1})P-\sum_{j=1}^{k}(s_j-s_{j-1})Q_{n_j}\right.\\
&\qquad\qquad\left.-\sum_{j,l}(s_j-s_{j-1})(s_l-s_{l-1})Q_{n_j}u_+^{-1}u_-^{-1}Qu_-u_+P\right).
\end{aligned}$$

Clearly,

$$Q_{n_j}u_+^{-1}u_-^{-1}Qu_-u_+P=Q_{n_j}a^{-1}QaP=:AP.$$

Since

$$\sum_{j=1}^{k}(s_j-s_{j-1})P-\sum_{j=1}^{k}(s_j-s_{j-1})Q_{n_j}=s_kP-\sum_{j=1}^{k}s_jP_{[n_j,n_{j+1})}=\sum_{j=0}^{k-1}(s_k-s_j)P_{[n_j,n_{j+1})}$$

and

$$\sum_{l}(s_l-s_{l-1})=s_k,\quad \sum_{j=1}^{k}s_k(s_j-s_{j-1})Q_{n_j}=\sum_{j=1}^{k}s_ks_jP_{[n_j,n_{j+1})},$$

we obtain

$$\det\left(I+\sum_{j=1}^{k}(s_j-s_{j-1})(P-aQ_{n_j}a^{-1})\right)$$

$$
\begin{aligned}
&= \det\left(I + \sum_{j=0}^{k-1}(s_j - s_k)P_{[n_j,n_{j+1})} - \sum_{j=1}^{k} s_k s_j P_{[n_j,n_{j+1})}AP\right)\\
&= \det\left(I + \sum_{j=0}^{k-1}(s_k - s_j)P_{[n_j,n_{j+1})}\right)\\
&\qquad \times \det\left(I - \left(\sum_{j=0}^{k}\frac{1}{1+s_k-s_j}P_{[n_j,n_{j+1})}\right)\left(\sum_{j=1}^{k} s_k s_j P_{[n_j,n_{j+1})}AP\right)\right)\\
&= \left(\prod_{j=0}^{k-1}(1+s_k-s_j)^{n_{j+1}-n_j}\right)\det\left(I - \left(\sum_{j=1}^{k}\frac{s_k s_j}{1+s_k-s_j}P_{[n_j,n_{j+1})}\right)A\right). \quad \blacksquare
\end{aligned}
$$

In [1] it is also shown that if a is a scalar-valued function without zeros on the unit circle and with winding number w, then (22) is true with the additional factor $(1+s_k)^w$ on the right-hand side. This can again be verified with the methods developed here, but we stop at this point.

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References

[1] J. Baik, P. Deift, and E.M. Rains: A Fredholm determinant identity and the convergence of moments for random Young tableaux. arXiv: math.CO/0012117.

[2] E.L. Basor and H. Widom: On a Toeplitz determinant identity of Borodin and Okounkov. *Integral Equations Operator Theory* **37** (2000), 397-401.

[3] A. Borodin and A. Okounkov: A Fredholm determinant formula for Toeplitz determinants. *Integral Equations Operator Theory* **37** (2000), 386-396.

[4] A. Böttcher: One more proof of the Borodin-Okounkov formula for Toeplitz determinants. arXiv: math.FA/0012200.

[5] A. Böttcher and B. Silbermann: *Analysis of Toeplitz Operators.* Springer-Verlag, Berlin, Heidelberg, New York 1990.

[6] H. Widom: Perturbing Fredholm operators to obtain invertible operators. *J. Funct. Analysis* **20** (1975), 26-31.

[7] H. Widom: Asymptotic behavior of block Toeplitz matrices and determinants II. *Adv. Math.* **21** (1976), 1-29.

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